Generating standing and propagating ocean waves with three-dimensional ARMA model

Technical report

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Section 1

Two methods of finding wave’s ACF

Analytic method

Apply Wiener—Khinchin theorem to a wave profile ζ to get ACF K:

$$K(t) = \mathcal{F}\left\{|\zeta(t)|^2\right\}.$$

Analytic method

Example

Standing wave profile:

$$\zeta(t, x, y) = A \sin(k_x x + k_y y) \sin(\sigma t).$$

Standing wave ACF:

$$K(t, x, y) = \gamma \exp\left[-\alpha(|t| + |x| + |y|)\right] \cos \beta t \cos\left[\beta x + \beta y\right].$$

Analytic method

Example

Propagating wave profile:

$$\zeta(t, x, y) = A \cos(\sigma t + k_x x + k_y y).$$

Propagating wave ACF:

$$K(t, x, y) = \gamma \exp\left[-\alpha(|t| + |x| + |y|)\right] \cos\left[\beta(t + x + y)\right].$$

Analytic method

Some observations:

- Taking Fourier transform of sine/cosine wave profile requires multiplying it by an decaying exponent to produce useful ACF.
- Fourier Transform of squared exponent (Gaussian) is another Gaussian.

Why use Fourier transform at all?

Empirical method

The algorithm:

1. Multiply wave profile by an decaying exponent.
2. Adjust sine/cosine phase to move maximum value to the origin (or substitute sine with cosine to get the same effect).

In case of plain waves result is the same as for analitic method.

Section 2

Governing equations for 3-dimensional ARMA process

3-D ARMA process

Three-dimensional autoregressive moving average process is defined by

$$\zeta_{i,j,k} = \sum_{l=0}^{p_1} \sum_{m=0}^{p_2} \sum_{n=0}^{p_3} \Phi_{l,m,n} \zeta_{i-l,j-m,k-n} + \sum_{l=0}^{q_1} \sum_{m=0}^{q_2} \sum_{n=0}^{q_3} \Theta_{l,m,n} \epsilon_{i-l,j-m,k-n},$$

where ζ — wave elevation, Φ — AR coefficients, Θ — MA coefficients, ϵ — white noise with Gaussian distribution, (p_1, p_2, p_3) — AR process order, (q_1, q_2, q_3) — MA process order, and $\Phi_{0,0,0} \equiv 0$, $\Theta_{0,0,0} \equiv 0$.

Determining coefficients

AR process

Solve linear system of equations (3-D Yule—Walker equations) for Φ:

$$\Gamma \begin{bmatrix} \Phi_{0,0,0} \\ \Phi_{0,0,1} \\ \vdots \\ \Phi_{p_1,p_2,p_3} \end{bmatrix} = \begin{bmatrix} K_{0,0,0} - \sigma_\epsilon^2 \\ K_{0,0,1} \\ \vdots \\ K_{p_1,p_2,p_3} \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{p_1} \\ \Gamma_1 & \Gamma_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Gamma_1 \\ \Gamma_{p_1} & \cdots & \Gamma_1 & \Gamma_0 \end{bmatrix},$$

$$\Gamma_i = \begin{bmatrix} \Gamma_i^0 & \Gamma_i^1 & \cdots & \Gamma_i^{p_2} \\ \Gamma_i^1 & \Gamma_i^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Gamma_i^1 \\ \Gamma_i^{p_2} & \cdots & \Gamma_i^1 & \Gamma_i^0 \end{bmatrix} \qquad \Gamma_i^j = \begin{bmatrix} K_{i,j,0} & K_{i,j,1} & \cdots & K_{i,j,p_3} \\ K_{i,j,1} & K_{i,j,0} & \ddots & x \quad \vdots \\ \vdots & \ddots & \ddots & K_{i,j,1} \\ K_{i,j,p_3} & \cdots & K_{i,j,1} & K_{i,j,0} \end{bmatrix}.$$

Determining coefficients

MA process

Solve non-linear system of equations for Θ:

$$K_{i,j,k} = \left[\sum_{l=i}^{q_1} \sum_{m=j}^{q_2} \sum_{n=k}^{q_3} \Theta_{l,m,n} \Theta_{l-i,m-j,n-k} \right] \sigma_\epsilon^2$$

via fixed-point iteration method:

$$\theta_{i,j,k} = -\frac{K_{0,0,0}}{\sigma_\epsilon^2} + \sum_{l=i}^{q_1} \sum_{m=j}^{q_2} \sum_{n=k}^{q_3} \Theta_{l,m,n} \Theta_{l-i,m-j,n-k}.$$

Determining coefficients

ARMA process

To mix processes one needs to divide ACF between processes, and recompute one of the parts to match process properties (mean, variance etc.).

There is no recomputation formula for 3-D proccess.

Our approach

Use AR process for standing waves and MA process for propagating waves.

Supporting experimental results:

- It works that way in practice.
- It does not work the other way round (processes diverge).
- Wavy surface integral characteristics match the ones of real ocean waves.

Section 3

Evaluation and verification

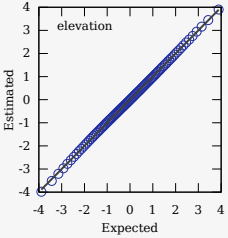
Experiment setup

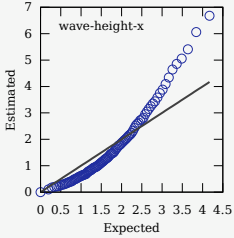
- Generate standing/propagating waves with AR/MA processes respectively.
- Estimate distributions of integral characteristics.
- Compare estimated distributions to the known ones via QQ plots.

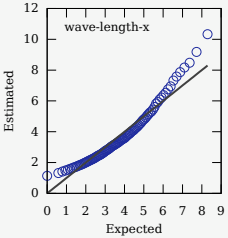
Characteristic	Weibull shape (k)
Wave height	2
Wave length	2.3
Crest length	2.3
Wave period	3
Wave slope	2.5
Three-dimensionality	2.5

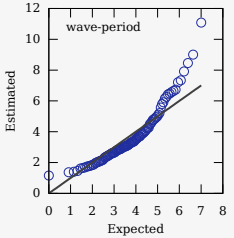
Verification results (QQ plots)

Standing waves









Propagating waves